Lesson 2-6 Quadratic Models

Vocabulary

quadratic model
quadratic regression

BIG IDEA Quadratic models are appropriate to consider when you think data will increase to a peak and then drop, or when data will decrease to a low point and then come back.

Mental Math

The identity $(x + y)^2 = x^2 + 2xy + y^2$ shows how to calculate $(x + 1)^2$ from x^2. Use it to find:

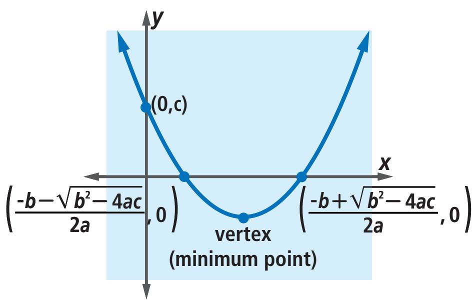
a. 41^2, since $40^2 = 1600$.

b. 101^2.

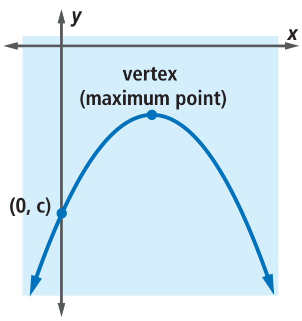
Linear functions are appropriate for modeling situations involving a constant amount of change. Exponential functions model situations involving a constant percent of change. In this lesson, we focus on **quadratic models**, that is, models based on quadratic functions.

Properties of Quadratic Functions

A *quadratic function* is a function with an equation that can be put into the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. Recall that the graph of a quadratic function is a *parabola*. If $a > 0$, the parabola opens up and has a *minimum point*, as shown at the left below. If $a < 0$, the parabola opens down and has a *maximum point*, as shown at the right below. The existence of these extrema distinguishes quadratic functions from linear and exponential functions.



$f(x) = ax^2 + bx + c$
$a > 0$
$b^2 - 4ac > 0$



$f(x) = ax^2 + bx + c$
$a < 0$
$b^2 - 4ac < 0$

The domain of a quadratic function is the set of all real numbers. When $a < 0$, the range is the set of all real numbers less than or equal to the maximum value. When $a > 0$, the range is the set of all real numbers greater than or equal to the minimum value. The y-intercept is the y-coordinate of the point where $x = 0$:

$$f(0) = a \cdot 0^2 + b \cdot 0 + c = c.$$

So, regardless of the value of a, c is the y-intercept.

The x-intercepts are the x-coordinates of the points where $y = 0$. The x-intercepts exist only when $b^2 - 4ac \geq 0$, and then can be found by solving the quadratic equation $ax^2 + bx + c = 0$. From the Quadratic Formula, the x-intercepts are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The maximum or minimum point of any quadratic function occurs at the x-value that is the mean of the solutions to the equation $f(x) = 0$, that is, when $x = -\frac{b}{2a}$.

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Example 1

Consider the function f with equation $f(x) = 2x^2 - 3x - 2$.

a. Find the x- and y-intercepts of its graph.

b. Tell whether the parabola has a maximum or minimum point, and find its coordinates.

Solution 1

a. Since f(0) = __?__, the y-intercept is -2.

To find the x-intercepts, let $f(x) = 0$ and solve for x.

$$2x^2 - 3x - 2 = 0$$

$$x = \frac{\underline{?} \pm \sqrt{\underline{?}^2 - 4 \cdot \underline{?} \cdot \underline{?}}}{2 \cdot \underline{?}}$$

$$= \frac{\underline{?} \pm \sqrt{\underline{?}}}{\underline{?}}$$

$$= 2 \text{ or } -\frac{1}{2}$$

The x-intercepts are 2 and $-\frac{1}{2}$.

b. Because the coef?cient of x^2 is __?__, the vertex is a minimum point. Since parabolas are symmetric, the x-coordinate of the minimum (or maximum) point occurs at the mean of the two x-intercepts.

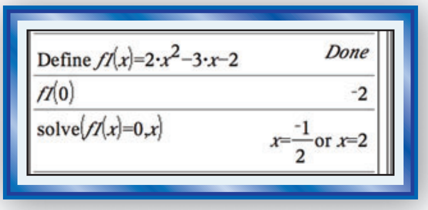
$$\frac{2 + \left(-\frac{1}{2}\right)}{2} = \underline{?}$$

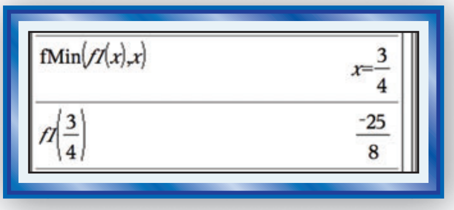
$$f(\underline{?}) = 2(\underline{?})^2 - 3(\underline{?}) - 2 = -\frac{50}{16} = -\frac{25}{8}$$

So the minimum point is $\left(\frac{?}{?}, -\frac{25}{8}\right)$, or (0.75, -3.125).

Solution 2

Use a CAS to find the x- and y-intercepts.





STOP QY

> **QY**
> What is the domain of the function in Example 1?

Using Known Quadratic Models

In the 17th century, extending earlier work of Galileo, Isaac Newton showed that the height h of an object at time t after it has been thrown with an initial velocity v_0 from an initial height h_0 satisfies the formula

$$h = -\frac{1}{2}gt^2 + v_0t + h_0,$$

where g is the acceleration due to gravity. Recall that velocity is the rate of change of distance with respect to time; it is measured in units such as miles per hour or meters per second. Acceleration is the rate at which velocity changes, so it is measured in units such as miles per hour *per hour* or meters per second^2. Near the surface of Earth, g is approximately $32\ \frac{\text{ft}}{\text{sec}^2}$ or $9.8\ \frac{\text{m}}{\text{sec}^2}$.

Taking a dive
The world record for the highest dive is 53.90 meters. It took place in Villers-le-Lac, France.

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Example 2

A ball is thrown upward from a height of 15 m with initial velocity 20 $\frac{\text{m}}{\text{sec}}$.

a. Find the relation between height h and time t after the ball is released.

b. How high is the ball after 3 seconds?

c. When will the ball hit the ground?

Solution

a. The conditions satisfy Newton's equation. Here $v_0 = \underline{\ ?\ }\ \frac{\text{m}}{\text{sec}}$, and $h_0 = \underline{\ ?\ }$ m. Use the metric system value $g = 9.8$.

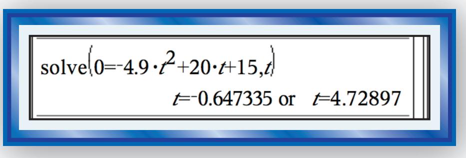
$$h = -\frac{1}{2}(\underline{\ ?\ })t^2 + \underline{\ ?\ }t + 15$$
$$h = -4.9t^2 + 20t + 15$$

b. Here $t = 3$ and you are asked to find h.

$$h = -4.9(3)^2 + 20 \cdot 3 + 15 = \underline{\ ?\ }$$

After 3 seconds, the ball is $\underline{\ ?\ }$ meters high.

c. At ground level, $h = \underline{\ ?\ }$. Solve $0 = -4.9t^2 + 20t + 15$ for t. Use a CAS to get $t \approx \underline{\ ?\ }$ or $t \approx \underline{\ ?\ }$. The negative value of t does not make sense in this situation, so we use the positive value. The ball will hit the ground after $\underline{\ ?\ }$ seconds.

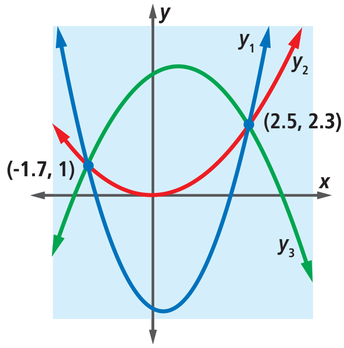


Finding the Quadratic Model through Three Points

The picture at the right shows that two points do not determine a parabola. To compute a unique quadratic model, you need a minimum of three noncollinear points.

One way to fit a quadratic model to data is to identify specific points on the model and set up a system of equations. The system must allow you to solve for the values of a, b, and c in the equation $y = ax^2 + bx + c$.

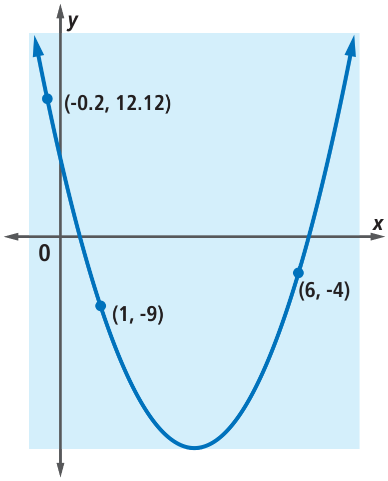
Alternately, you can use *quadratic regression*, a calculation available in many statistics packages. **Quadratic regression** is a technique, similar to the method of least squares, that finds an equation for the best-fitting parabola through a set of points.



There are many parabolas that pass through the two points (-1.7, 1) and (2.5, 2.3).

Example 3

The parabola at the right contains points (1, –9), (6, –4), and (–0.2, 12.12). Find its equation.



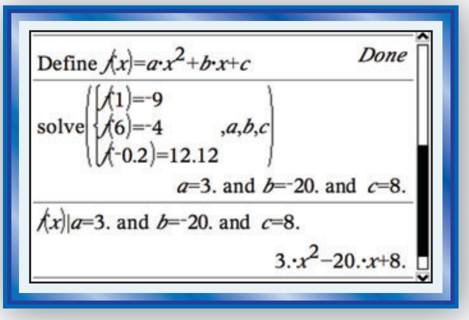
Solution 1 Because the ordered pairs (x, y) are solutions of the equation $y = ax^2 + bx + c$, substitute to get 3 linear equations, each with a, b, and c as unknowns.

$$\begin{aligned} f(1) &= -9 = a(1)^2 + b(1) + c \\ f(6) &= -4 = a(6)^2 + b(6) + c \\ f(-0.2) &= 12.12 = a(-0.2)^2 + b(-0.2) + c \end{aligned}$$

This produces a system of three equations.

$$\begin{cases} -9 = a + b + c \\ -4 = 36a + 6b + c \\ 12.12 = 0.04a - 0.2b + c \end{cases}$$

Use the `solve` command on a CAS. An equation for the parabola that contains these three points is $f(x) = 3x^2 - 20x + 8$.



Solution 2 Use quadratic regression. You are asked to do this in Question 8.

Check Substitute the points into the equation.

Does $3(1)^2 - 20(1) + 8 = -9$? Yes.
Does $3(6)^2 - 20(6) + 8 = -4$? Yes.
Does $3(-0.2)^2 - 20(-0.2) + 8 = 12.12$? Yes, it checks.

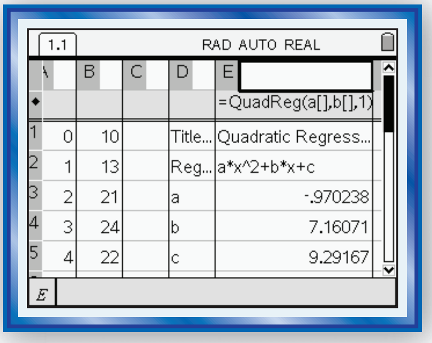
When data points all lie on a single parabola, as in Example 3, the system strategy will yield an exact model. The model found by solving a system will be identical to the model formed by using quadratic regression. However, if the data show a quadratic trend, but not an exact quadratic fit, then the two solution strategies may yield slightly different equations.

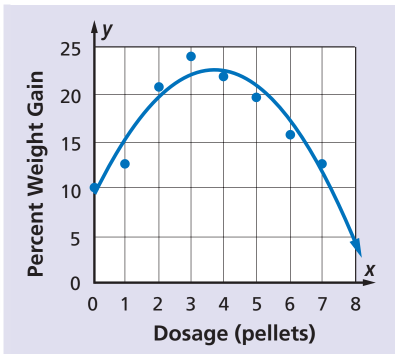
Fitting a Quadratic Model through More Than Three Points

The following table contains data that might be collected by farmers interested in increasing the weight of their pigs. Suppose twenty-four randomly selected pigs were each given a daily dosage (in pellets) of a food supplement. Each group of three pigs received a dosage from 0 to 7 pellets, and the average percent weight gain for each group was recorded. The table below shows the average percent weight gain for each group of three pigs in relation to the number of pellets they were given daily.

Dosage (pellets)	0	1	2	3	4	5	6	7
Percent Weight Gain	10	13	21	24	22	20	16	13

The data show that more is not necessarily better. The pigs' bodies start rejecting the supplement when the dosage is too high. So there is a peak in the data and a quadratic model might be appropriate. A scatterplot of the data and the graph of the quadratic regression model $y = -1.0x^2 + 7.2x + 9.3$ are shown below. With the exception of the point (3, 24), the data points lie fairly close to the parabola.





There is something quite different about this application when compared to the application in Example 2. There is no theory that links dosage with percent weight gain as there is with projectile motion. Models such as this one are called *impressionistic models* or *non-theory-based models*, because no theory exists that explains why the model fits the data. This is different from Example 2, where the well-established theory of gravity and all sorts of real data have verified that the height of a projectile is a quadratic function of time.

Questions

COVERING THE IDEAS

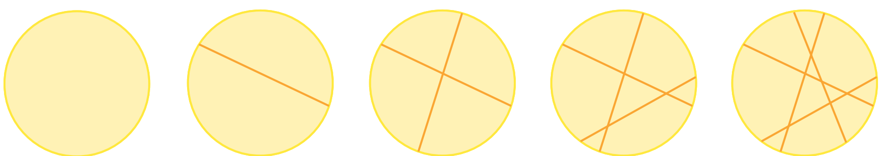
1. What is the general form of an equation of a quadratic function?
2. What values of x are the solutions to $ax^2 + bx + c = 0$?
3. What is the range of the function f in Example 1?
4. Consider the graph of the function f with $f(x) = 2x^2 - x - 4$.
   a. Give its y- and x-intercepts.
   b. Sketch the part of the graph where $-3 \le x \le 3$.
   c. Give the coordinates of the minimum point.
5. Tell whether the graph of the equation has a maximum point, a minimum point, or neither.
   a. $y = 8x^2 - 3x - 7$
   b. $y = 2x + 4x^2$
   c. $y = 6 - 2x^2$
   d. $y = -x^2 + 5x + 177$
6. Repeat Example 2 as if the ball were on the moon. Acceleration due to gravity on the moon is $1.6 \frac{\text{m}}{\text{s}^2}$.

In 7 and 8, refer to Example 3.

7. Using the quadratic model, show that $f(4) = -24$.
8. Find the quadratic model by using quadratic regression.
9. Refer to the data about weight gain in pigs on page 121.
   a. What does the model predict for the percent weight gain for pigs fed 4.5 pellets daily?
   b. Is the prediction in Part a extrapolation or interpolation?
10. A parabola contains the points (0, 1), (4, 5), and (8, 7).
    a. Graph these points and estimate the coordinates of the vertex.
    b. Find an equation for the parabola by setting up and solving a system of equations.
    c. Check your estimate in Part a.

APPLYING THE MATHEMATICS

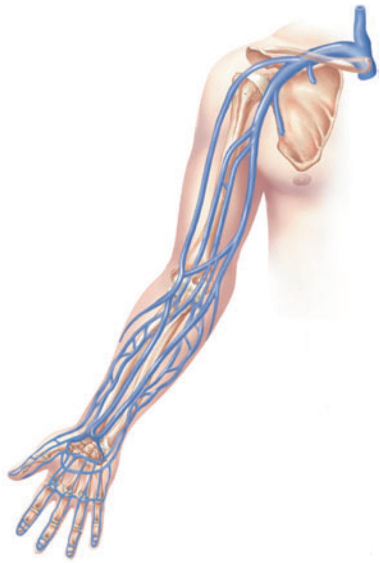
11. The table below shows the largest number of pieces $f(n)$ into which a pizza can be cut by n straight cuts.



n	0	1	2	3	4
$f(n)$	1	2	4	7	11

    a. Fit a quadratic model to these data using regression.
    b. Use your model to find the greatest number of pieces produced by 5 straight cuts. Check your answer by drawing a diagram.

12. A piece of an artery or a vein is approximately the shape of a cylinder. The French physiologist and physician Jean Louis Poiseuille (1799–1869) discovered experimentally that the velocity v at which blood travels through arteries or veins is a function of the distance r of the blood from the axis of symmetry of the cylinder. For example, for a wide arterial capillary, the following formula might apply: $v = 1.185 - (185 \cdot 10^4)r^2$, where r is measured in cm and v in $\frac{\text{cm}}{\text{sec}}$.
   a. Find the velocity of blood traveling on the axis of symmetry of this capillary.
   b. Find the velocity of blood traveling $6 \cdot 10^{-4}$ cm from the axis of symmetry.
   c. According to this model, where in the capillary is the velocity of the blood 0?
   d. What is the domain of the function mapping r onto v?
   e. Sketch a graph of this function.



13. Use the table at the right showing the amount of bar iron exported to England from the American Colonies from 1762 to 1774. Bar iron is measured in "old" tons of 2240 pounds.
   a. Construct a scatterplot for these data with the independent variable as years after 1762.
   b. Find the quadratic regression model for these data.
   c. Use your quadratic model to predict the amount of bar iron exported in 1776. (The actual value was 28 old tons.)
   d. Why is extrapolation to 1776 inappropriate?

Year	Bar Iron Exported (old tons)
1762	110
1763	310
1765	1079
1768	1990
1770	1716
1771	2222
1773	838
1774	639

Source: U.S. Census Bureau

14. The Center for Disease Control studies trends in high school smoking. The percent of students in grades 9 through 12 who reported smoking cigarettes on 20 of the 30 days preceding the administration of the National Youth Risk Behavior Survey (frequent cigarette use) increased in the 1990s, but decreased after 1999.

Years after 1990	1	3	5	7	9	11	13	15
% of Frequent Cigarette Use = y	12.7	13.8	16.1	16.7	16.8	13.8	9.7	9.4

   a. Construct a scatterplot for these data.
   b. Calculate the sum of squared residuals for the quadratic model $y = -0.2x^2 + 2x + 11$.
   c. Find the quadratic regression model for the data. Calculate the sum of squared residuals for this model.
   d. Which model has a smaller sum of squared residuals?
   e. Using the regression model, predict the cigarette use in 2006 and 2010. Do you think the predictions are reasonable?

REVIEW

15. The half-life of Th-232 (Thorium) is 14.05 billion years. Suppose a sample contains 100 grams of pure thorium. **(Lesson 2-5)**

a. Find an equation for the amount of Th-232 left as a function of the number of billions of years that have passed.

b. Find the amount left after 13.7 billion years, the estimated age of our atmosphere.

In 16 and 17, Sri Lanka and Madagascar are two island nations in the Indian Ocean. In 2007, the population of Sri Lanka was about 20.9 million people and the population of Madagascar was about 19.4 million people. In 2007, the population of Sri Lanka was growing at a rate of about 0.98% annually, while the population of Madagascar was growing at a rate of about 3.01% annually. Let the function $S(x) = ab^x$ represent the population (in millions) of Sri Lanka x years after 2007, and let the function $M(x) = cd^x$ represent the population (in millions) of Madagascar x years after 2007.

Kandy

A town in the center of Sri Lanka

16. a. Determine a, b, c, and d and write formulas for $S(x)$ and $M(x)$.

b. Assuming constant growth rates, use your formulas from Part a to estimate the populations of Sri Lanka and Madagascar in 1997.

c. Graph $y = S(x)$ and $y = M(x)$ on the same set of axes for $0 \le x \le 50$.

d. Make a prediction about how the future populations of Sri Lanka and Madagascar will compare if current trends continue. **(Lesson 2-4)**

17. When the function S is used to predict the population of Sri Lanka in 2008, the residual is 2.848. What was the observed population? **(Lesson 2-2)**

18. What are the maximum and minimum possible values for a correlation coefficient? **(Lesson 2-3)**

19. ***Skill Sequence*** Solve for x. **(Previous Course)**

a. $(5x - 10)(x - 3) = 0$ **b.** $3x^2 = 1 - 2x$ **c.** $3x^4 = 1 - 2x^2$

20. Sketch the graph of $y = \frac{1}{x}$. **(Previous Course)**

EXPLORATION

21. a. Find the vertices for the family of quadratic functions $y = x^2 + bx + 1$ for $b = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

b. Graph all of your collected vertices in a scatterplot.

c. Find an exact quadratic model for the data.

d. If you had started this question with $y = 2x^2 + bx + 1$, what would you predict the quadratic model of the vertex data to be? Why?

QY ANSWER

$\mathbb{R}$